Metalogic for Sentential Logic

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Contents

Logic and Metalogic

- Logic studies **logical consequence** and **logical truth**.
- "Logic is the science of correct argument"(Vann McGee)

Logic and Metalogic

- Historical figures: Frege, Russell, Tarski, Gentzen,...
- Sometimes "logic" is used in restricted sense to refer only to proof procedure, but I will use the term in broad sense, including semantic system.
- Also, "logic" has had various meanings in different context. But basically the term should be used in relevant sense to **formal** or **deductive** logic.
- Inductive logic is also a part of logic but usually a subject of philosophy of science.

Logic and Metalogic

- Metalogic deals with questions *about* formal system: Is the system sound or complete? Is this formula derivable from the system? and so on.
- In elementary course, we learn how to *use* formal system. In metalogic, we learn something *about* the formal system.
- KEEP IN MIND: USE AND MENTION DISTINCTION

Logic and Metalogic

- To return to the concept of "logical consequence" and "logical truth"
- There are two well-established characterization of the concepts: *semantic* and *proof-theoretic*
- "Semantic" suggests a relation between words and worlds. So semantic characterization of the concepts contains truth and falsity.
- "ϕ is logically true." means "ϕ is valid"
  "ϕ is logical consequence of Γ" means "ϕ is semantic consequence of Γ"

Logic and Metalogic

- By contrast, “proof-theoretic” rather suggests syntactic characterization. It only deals with the form of the given formulas.
- “ϕ is logically true.” means “ϕ is a theorem”
  “ϕ is logical consequence of Γ” means “ϕ is derivable from Γ”
- At the final step, we will see how two characterizations can coincide.

Syntax for L

- We will build up a very simple language.
- It contains some non-logical symbols that represent propositions, logical connectives, and parentheses.

Vocabulary of formal language *L*

(1)	Sentential symbols	"A", "B", "C", ...
(2)	Logical connectives	"$\vee$", "$\sim$", "$\wedge$", "$\rightarrow$", "$\equiv$"
(3)	Parentheses	"(", ")"

Syntax for L

- Next, as ordinary languages, grammar should be set out. The following is the rules for generating grammatical sentences, called "well-formed formulas(wffs)."
- NOTE: "ϕ" and "ψ" are metalogical symbols, playing the role of variable for well-formed formulas. They are not part of this formal language.

Well-formed formulas of L

(1) All sentential symbols are well-formed formulas.
(2) If ϕ and ψ are well-formed formulas, then $\sim \phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \equiv \psi)$ are also well-formed formulas.
(3) Nothing except generated by above rules are well-formed formulas.

Semantics for L

- In sentential logic, to build semantic system is to assign truth-value to wffs. This is essentially same job when we make up **truth table** of formulas.
- What different from truth table method is that it is usually represented by *function.*
- We have two sort of functions.
  (1) On the one hand, there is *truth assignment function f* which assigns either 1 or 0 to each sentential symbols.
  (2) On the other hand, there is *valuation function with respect to f*, V_f which assigns 1 or 0 to each wffs by following rules:

Semantics for L

For any sentential symbol α and any wffs ϕ and ψ,

$V_f(\alpha) = f(\alpha)$,

$V_f(\sim \phi) = 1$ iff $V_f(\phi) = 0$,

$V_f(\phi \vee \psi) = 1$ iff $V_f(\phi) = 1$ or $V_f(\psi) = 1$,

$V_f(\phi \rightarrow \psi) = 1$ iff $V_f(\phi) = 0$ or $V_f(\psi) = 1$, ...

- Notice that there can be many different truth assignment functions(or *interpretation*). It is possible that under one interpretation "A" is assigned to 1, but under other interpretation it is assinged to 0.
- In truth table, each rows are possible interpretations and corresponding valuations.

Semantics for L

- Now we can define validity of wff.

  ϕ is valid($\models \phi$) if and only if for any interpretation f, $V_f(\phi) = 1$.

- And semantic consequence.

  ϕ is semantic consequence from the set of wff $\Gamma(\Gamma \models \phi)$ if and only if for any interpretation f, if $V_f(\gamma) = 1$ for all $\gamma \in \Gamma$, then $V_f(\phi) = 1$.

Proof Theory for L

- Proof does not care about truth-value of formulas.
  It only has to do with *forms of formulas* and its derivation by the inference rule.
- Eventually, the proof of a formula can be represented as *a sequence of formulas in which the proved formula is placed at the last line.*
- Two well-known proof systems: axiomatic proof, natural deduction.
- We will see the system of natural deduction presented by Irving Copi.

Proof Theory for L

전건긍정법(MP)	후건부정법(MT)	가정적 삼단논법(HS)
$p \to q$ p $\therefore q$	$p \to q$ $\sim q$ $\therefore \sim p$	$p \to q$ $q \to r$ $\therefore p \to r$
선언적 삼단논법(DS)	**연언지 단순화(Simp)**	**연언지 결합(Conj)**
$p \vee q$ $\sim p$ $\therefore q$	$p \wedge q$ $\therefore p$	p q $\therefore p \wedge q$
선언지 추가(Add)	**딜레마(CD)**	**흡수규칙(Abs)**
p $\therefore p \vee q$	$(p \to q) \wedge (r \to s)$ $p \vee r$ $\therefore q \vee s$	$p \to q$ $\therefore p \to (p \wedge q)$

(a) Implication Rules

드모르간 법칙	$\sim(p \vee q)$	::	$\sim p \wedge \sim q$
(DM)	$\sim(p \wedge q)$	::	$\sim p \vee \sim q$
교환 법칙	$p \vee q$	::	$q \vee p$
(Com)	$p \wedge q$	::	$q \wedge p$
배분 법칙	$p \wedge (q \vee r)$	::	$(p \wedge q) \vee (p \wedge r)$
(Dist)	$p \vee (q \wedge r)$	::	$(p \vee q) \wedge (p \vee r)$
결합 법칙	$p \vee (q \vee r)$	::	$(p \vee q) \vee r$
(Assoc)	$p \wedge (q \wedge r)$	::	$(p \wedge q) \wedge r$
이중 부정 (DN)	p	::	$\sim\sim p$
대우 (Trans)	$p \to q$	::	$\sim q \to \sim p$
단순 함축 (Impl)	$p \to q$	::	$\sim p \vee q$
단순 동치	$p \leftrightarrow q$	::	$(p \to q) \wedge (q \to p)$
(Equiv)	$p \leftrightarrow q$	::	$(p \wedge q) \vee (\sim p \wedge \sim q)$
수출입 규칙 (Exp)	$(p \wedge q) \to r$	::	$p \to (q \to r)$
동어반복	p	::	$p \vee p$
(Taut)	p	::	$p \wedge p$

(b) Substitution Rules

Proof Theory for L

- There is no axiom in this proof system. Given some premises, you can derive a conclusion by applying above rules.
- I have said that "ϕ is logically true." means "ϕ is a theorem," but I think that's not appropriate concept for this system. It is for axiomatic system.
- There are some proof techniques, called "Conditional Proof" and "Indirect Proof" which makes proof procedures easier but considering simplicity, they are just useful tools but not essential.
- Many on the list may turn out to be redundant by the corollary of completeness theorem.

Soundness and Completeness

- Here, “is sound” and “is complete” are predicates that have a logical system as its subject.
- Simply the soundness theorem states that **all derivable formulas are valid.**
- The completeness theorem states that **all valid formulas are derivable.**

Soundness and Completeness

- Each has two senses, ‘weak’ and ‘strong’ but we will going to deal only with ‘strong’ one.
- Indeed, soundness theorem is quite trivial. Because our natural deduction rules are designed to produce valid formulas.
- Of course the theorem itself is not trivial normally. But even in that case, it is much easier to prove rather than to prove completeness theorem.

Soundness and Completeness

- Now let's prove that our sytem is complete.
  Our goal is to prove:

  If $(P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology, then Q is derivable from $P_1, P_2, P_3, ..., P_n$.

- As to the antecedent of the theorem, it is logically equivalent to the proposition that Q is semantic consequence of $P_1, P_2, P_3, ..., P_n$. Following equivalence is hold.

  ϕ logically implies ψ if and only if $(\phi \rightarrow \psi)$ is a tautology.

Soundness and Completeness

- In advance, following is taken to be granted

  T1. For any sentential schema S which is a tautology, there is a conjunctive normal form of it.

- Conjuntive normal form is such that every conjunct is a disjunction that contains some sentential variables together with its negation.
- We can obtain a conjunctive normal form of S, say CNF(S), by applying several substitution rules.

Soundness and Completeness

- **T2.** $\phi \vee \sim \phi$ **is derivable from any non-empty premise set.**

  Let $P_1, P_2, ..., P_n (n > 0)$ be given premises.

1.	P_1	Assumption
2.	...	Assumption
n.	P_n	Assumption
n+1.	$P_1 \vee \sim \phi$	n, Add.
n+2.	$\phi \rightarrow P_1$	n+1, Comm, Impl.
n+3.	$\phi \rightarrow (\phi \wedge P_1)$	n+2, Abs.
n+4.	$\sim \phi \vee (\phi \wedge P_1)$	n+3, Impl.
n+5.	$(\sim \phi \vee \phi) \wedge (\sim \phi \vee P_1)$	n+4, Dist.
n+6.	$\phi \vee \sim \phi$	n+5, Simp, Comm.
		$\square$

Soundness and Completeness

- **T3. Let** $Q = (Q_1 \vee Q_2 \vee ... \vee \phi \vee \sim \phi \vee ... \vee Q_m)$ **be any sentential schemata, then from a sequence** $P_1, P_2, P_3, ..., P_n$**,** Q **is derivable.**

  Since $\phi \vee \sim \phi$ is derivable from $P_1, P_2, P_3, ..., P_n$(by T2), one can derive Q from it by iteratively applying Add. and Comm. rules.

- **T4. Let S be any tautology. Then CNF(S) is derivable from** $P_1, P_2, P_3, ..., P_n$**.**

  By T3, any form of Q is derivable from $P_1, P_2, P_3, ..., P_n$, every conjunct of CNF(S) is derivable. And then by applying Conj rule repeatedly, CNF(S) can be derived from $P_1, P_2, P_3, ..., P_n$.

Soundness and Completeness

- **T5. Let S be any tautology. Then S is derivable from $P_1, P_2, P_3, ..., P_n$.**

  By T4, CNF(S) for any tautology S is derivable from $P_1, P_2, P_3, ..., P_n$. And by T1, from CNF(S) S can be derived, so consequently S is derivable from $P_1, P_2, P_3, ..., P_n$.

- **Completeness: If $(P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology, then Q is derivable from $P_1, P_2, P_3, ..., P_n$**

  Suppose that $(P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology. Then by T5, it is derivable from $P_1, P_2, P_3, ..., P_n$. And by repeating Conj rule, $(P_1 \wedge P_2 \wedge P_3 \wedge ... \wedge P_n)$ is also derivable from the very same set. By *Modus Ponens*, Q is derivable from $P_1, P_2, P_3, ..., P_n$. $\square$

Soundness and Completeness

- There are some rules we didn't use in proving the theorem.
  From the fact, it follows that the unused rules are redundant to the completeness of our system.
- The significance of both theorems are important for mathematicians or who studies logical systems.
  If two theorems hold for a system, it means that if the system found that a formula is logically true, then it would be able to prove it, and vice versa.

Thank you!